



Fracture Mechanics

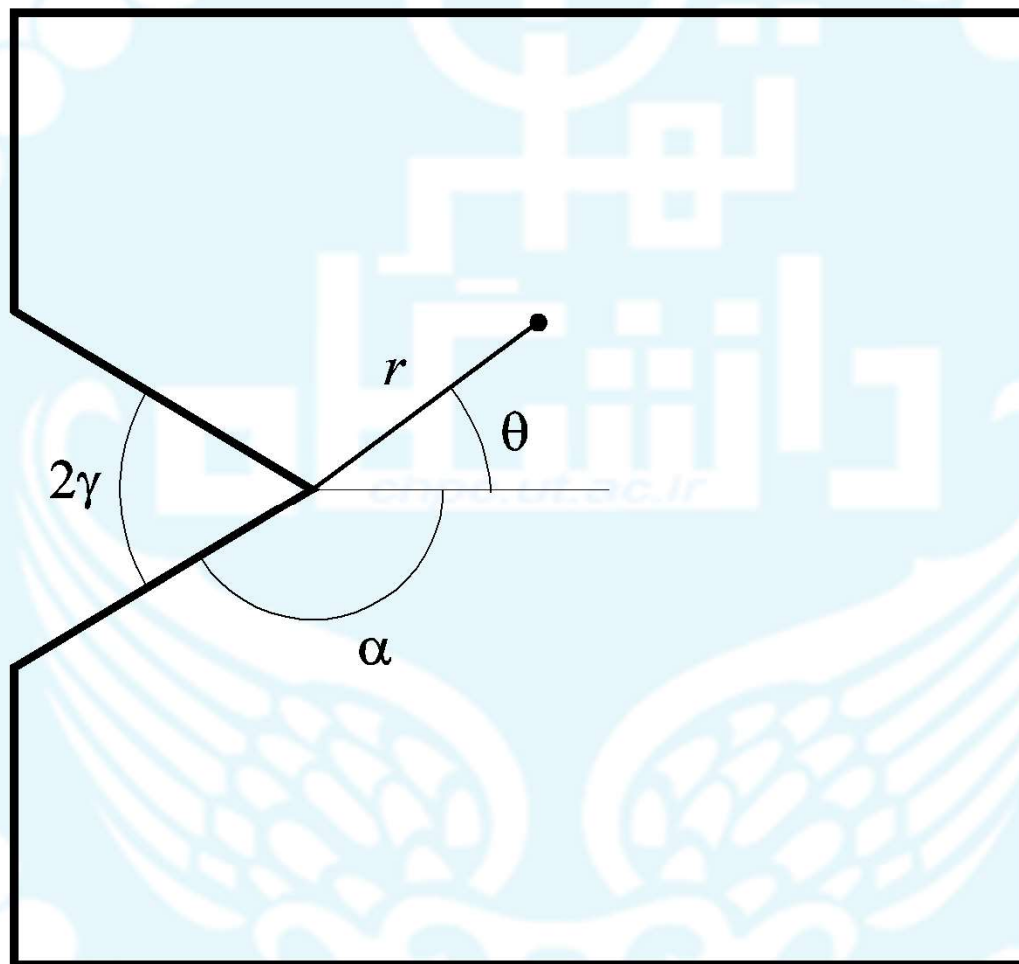
Soheil Mohammadi
School of Civil Engineering
University of Tehran

Tel: +98-21-6111 2258, Fax: +98-21-6640 3808, Cell: 0912-5030 392

Email: smoham@ut.ac.ir, Homepage: <http://chpc.ut.ac.ir/smohammadi>
Facebook: [prof.soheil.mohammadi](https://www.facebook.com/prof.soheil.mohammadi)

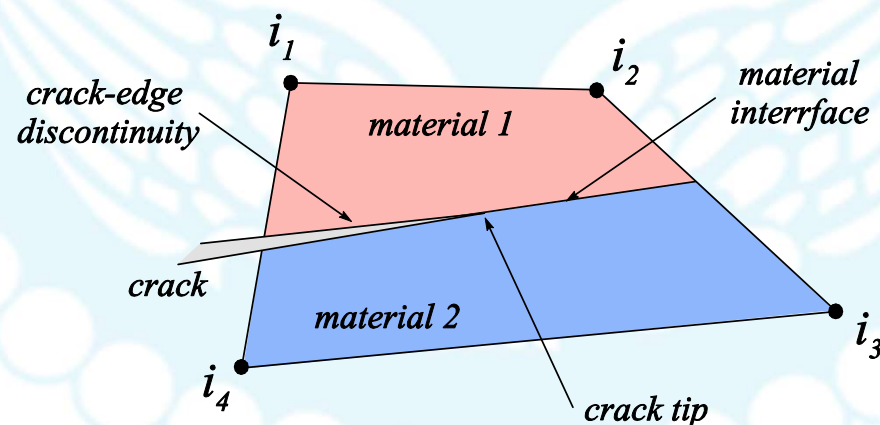


Williams Solution



Williams (1952) proposed a new set of solutions for a plate with angular corners (wedges), as depicted in Figure 2.6b. The method is capable of determining the stress and displacement fields near and far from a crack tip.

It can also be extended to a crack at the interface of two dissimilar materials, and even to a general solution for a contact stick/slip crack between dissimilar materials.



Williams' solution can be defined as,

$$\begin{aligned}\Phi(r, \theta) &= r^{\lambda+1} F(\theta, \lambda) = r^{\lambda+1} e^{m(\lambda)\theta} \\ &= r^{\lambda+1} [A \cos(\lambda-1)\theta + B \cos(\lambda+1)\theta + C \sin(\lambda-1)\theta + D \sin(\lambda+1)\theta]\end{aligned}$$

where

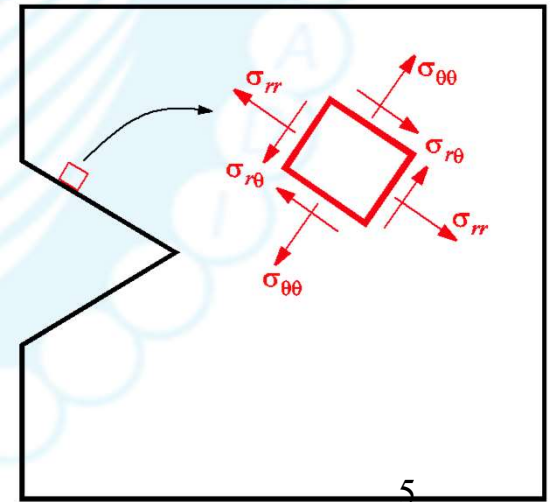
$$m(\lambda) = \pm i(\lambda \pm 1)$$

Imposing the traction-free boundary conditions at the corner edges,

$$\sigma_{\theta\theta}|_{\theta=\pm\alpha} = 0$$

$$\sigma_{r\theta}|_{\theta=\pm\alpha} = 0$$

results in the following characteristic equation



$$\sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial^2 r} = r^{\lambda-1} \lambda(\lambda+1) F(\theta)$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right) = r^{\lambda-1} [-\lambda F'(\theta)]$$

results in the following characteristic equation

$$\sin 2\lambda_n \alpha + \lambda_n \sin 2\alpha = 0$$

or for the simplified case of a sharp crack $\alpha = \pi$

$$\sin 2\pi\lambda_n = 0, \quad \lambda_n = \frac{n}{2}, \quad n = 1, 3, 4, \dots$$

The final polar components of the stress field for the crack problem is obtained as,

$$\sigma_{rr} = \frac{1}{\sqrt{r}} \sum_n \left\{ b_n \left[\cos \frac{\theta}{2} \left(1 + \sin^2 \frac{\theta}{2} \right) \right] + a_n \left[-\frac{5}{4} \sin \frac{\theta}{2} + \frac{3}{4} \sin \frac{3\theta}{2} \right] \right\}$$

$$\sigma_{\theta\theta} = \frac{1}{\sqrt{r}} \sum_n \left\{ b_n \left[\cos \frac{\theta}{2} \left(1 - \sin^2 \frac{\theta}{2} \right) \right] + a_n \left[-\frac{3}{4} \sin \frac{\theta}{2} - \frac{3}{4} \sin \frac{3\theta}{2} \right] \right\}$$

$$\sigma_{r\theta} = \frac{1}{\sqrt{r}} \sum_n \left\{ b_n \left[\sin \frac{\theta}{2} \cos^2 \frac{\theta}{2} \right] + a_n \left[\frac{1}{4} \cos \frac{\theta}{2} + \frac{3}{4} \cos \frac{3\theta}{2} \right] \right\}$$

$$b_n = \frac{C_n}{D_n} = -\frac{\sin(\lambda_n - 1)\pi}{\sin(\lambda_n + 1)\pi}, \quad \lambda_n = \frac{n}{2}, \quad n = 1, 3, 4, \dots$$

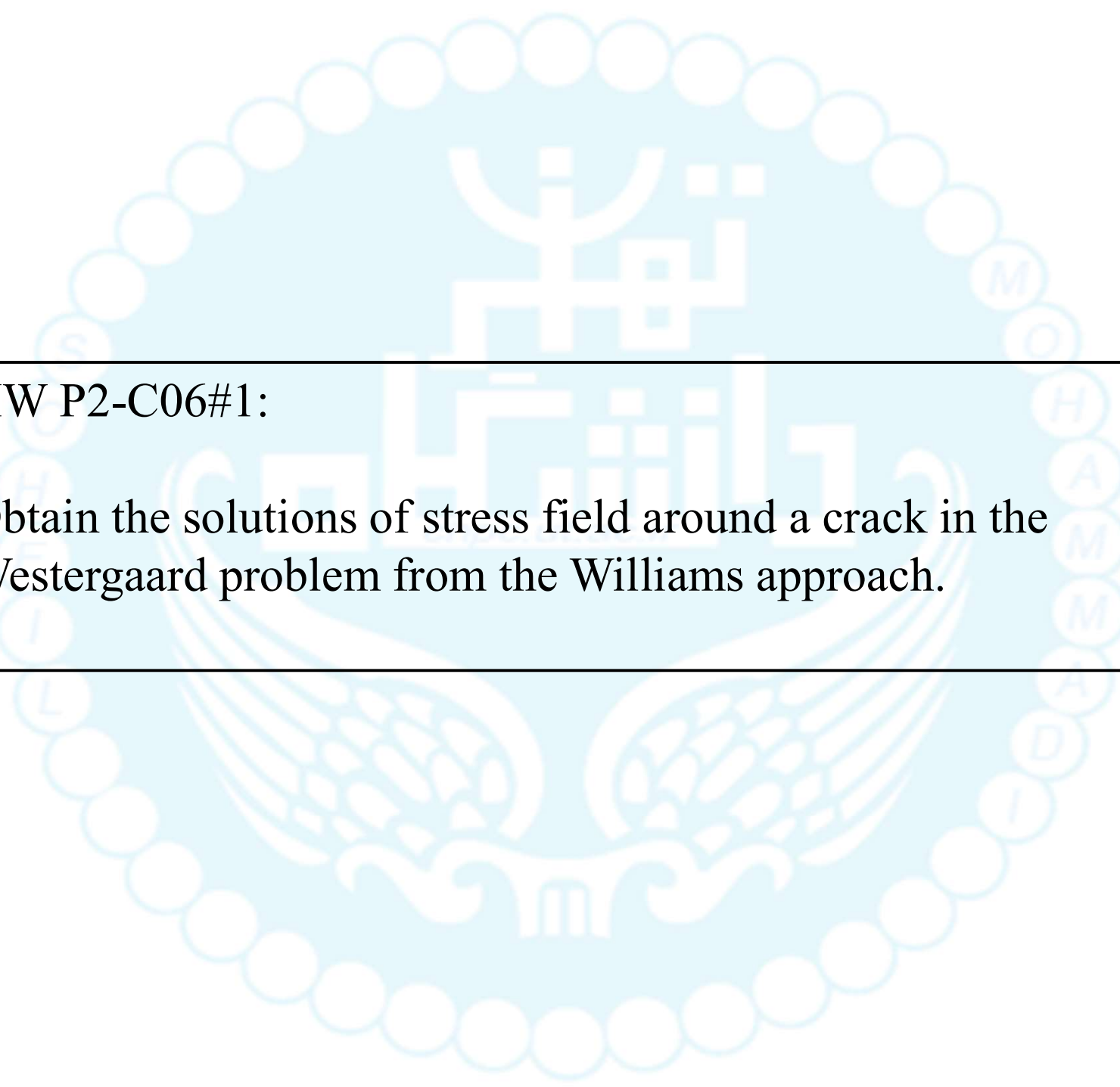
$$a_n = \frac{B_n}{A_n} = -\frac{\cos(\lambda_n - 1)\pi}{\cos(\lambda_n + 1)\pi}, \quad \lambda_n = \frac{n}{2}, \quad n = 1, 3, 4, \dots$$

These equations include both opening and shearing modes of crack response, associated with b_n and a_n terms, respectively.

The first term of the series ($\lambda_1 = 1/2$) represents the Westergaard approximate solution for the opening mode near the crack tip.

$$n=1 \quad \begin{aligned} b_1 &= \frac{K_I}{\sqrt{2\pi}} \\ a_1 &= \frac{K_{II}}{\sqrt{2\pi}} \end{aligned}$$

Similar radial and angular functions are obtained in both approaches. Several more terms, however, have to be computed in order to compute accurate solutions for points far from the crack tip.



HW P2-C06#1:

Obtain the solutions of stress field around a crack in the Westergaard problem from the Williams approach.

